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M.Sc. Sem II Paper 9th

TOPOLOGY (contd.)

Theorem Prove that a topological space X is normal iff for every closed set F and open set H containing F, there exists an open set G such that

$F \subseteq G \subseteq \overline{G} \subseteq H$.

Proof Necessary part

Let X be a normal space.

Let F be a closed set in X. Let H be an open set such that $F \subseteq H$.

we have to prove that $\exists$ an open set G ~~con~~ such that

$F \subseteq G \subseteq \overline{G} \subseteq H$.

$\because$ H is open $\Rightarrow H^c$ is closed.

$\because F \subseteq H \Rightarrow F \cap H^c = \phi$

So, F and H^c are disjoint closed sets in X. Also, X is a normal space.

So, by definition of normal space,

there exist open sets G and M such that

$F \subseteq G$, $H^c \subset M$, and $G \cap M = \phi$.

Now, $G \cap M = \phi \Rightarrow G \subseteq M^c$

But $H^c \subset M \Rightarrow M^c \subseteq H$.

Also, M is open $\Rightarrow M^c$ is closed.

Hence, $F \subseteq G \subseteq \bar{G} \subseteq M^c \subseteq H$.

Proved

Sufficient part Given that

$F \subset G \subset \bar{G} \subset H$, i.e. the given condition is satisfied.

We have to prove that X is normal.

Let F_1 and F_2 be any two disjoint closed sets in X.

$\Rightarrow F_1 \cap F_2 = \phi \Rightarrow F_1 \subset F_2^c$

$\because F_2$ is closed $\Rightarrow F_2^c$ is open.

So, $\exists$ an open set G such that

$F_1 \subseteq G \subset \bar{G} \subseteq F_2^c$.

Now, $\bar{G} \subseteq F_2^c \Rightarrow F_2 \subseteq \bar{G}^c$

and $G \subset \bar{G} \Rightarrow G \cap \bar{G}^c = \phi$ and $\bar{G}^c$ is open.

Thus, $F_1 \subseteq G$, $F_2 \subset \bar{G}^c$, G and $\bar{G}^c$ are disjoint open.

$\Rightarrow X$ is normal.

Proved.